

Finite Amplitude Convection in a Rotating Porous Medium

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Z. Naturforsch. **42a**, 13–20 (1987); received July 30, 1986

The onset of convection in a horizontal layer of a saturated porous medium heated from below and rotating about a vertical axis with uniform angular velocity is investigated. It is shown that when $S \in \sigma > 1$, overstability cannot occur, where ε is the porosity, σ the Prandtl number and S is related to the heat capacities of the solid and the interstitial fluid. It is also shown that for small values of the rotation parameter T_1 , finite amplitude motion with subcritical values of Rayleigh number R (i.e. $R < R_c$, where R_c is the critical Rayleigh number according to linear stability theory) is possible. For large values of T_1 , overstability is the preferred mode.

1. Introduction

Thermal convection in a fluid layer heated from below is one of the simplest examples for the instability mechanism of a fluid dynamical system, which has been and is still studied extensively both experimentally and theoretically (see Review article by Busse [1]). The corresponding flow in a saturated porous medium has also attracted considerable attention in the past, especially among geophysicists and chemical engineers. Onset of thermal instability in a nonrotating system of a porous medium heated from below was studied by Lapwood [2], Wooding [3, 4], and many others using linear stability theory. Elder [5] and Combarnous and Le Fur [6] determined experimentally the point at which thermal convection began and found good agreement with the theory. Gheorghita [7] extended Lapwood's theory to nonhomogeneous porous media. Dandapat and Gupta [8] extended this deterministic theory to the case of a horizontal layer subject to random vibrations. Palm, Weber, and Kvernfold [9] showed the nature of the dependence of the Nusselt number on the Rayleigh number in steady convection in a porous medium. However, natural convection in a rotating porous layer has not been given much attention so far. Using linear stability theory, Rudraiah and Srimani [10] investigated the thermal instability in a fluid-saturated porous layer which

rotates with a uniform angular velocity about a vertical axis. The above stability analysis was extended by Friedrich and Rudraiah [11] to include finite amplitude disturbances by using Stuart's shape function. Assuming stationary convection, they plotted the streamlines and isotherms for large Rayleigh numbers and also determined the dependence of the Nusselt number (characterizing heat flux) on the Rayleigh number.

However the analysis in [10] is confined to the case of a medium with the porosity ε having the value unity and with negligible viscous forces. It is well-known, however, that the viscous forces are important in a medium with $\varepsilon = 1$ when the fluid occupies the major portion of the porous medium. Further the criterion derived in [10] for the existence of overstability is dependent on the wave number, and hence this result is of limited interest.

In the first part of the paper we have carried out a linear stability analysis for the onset of convection in a rotating layer of a saturated porous medium with $\varepsilon \neq 1$, assuming that the viscous forces are not negligible inside the medium. It is shown that overstability cannot occur if $S\varepsilon\sigma > 1$, where σ is the Prandtl number and $S = \varepsilon + (1 - \varepsilon)(\rho c)_s/(\rho c)_f$ with $(\rho c)_s$ and $(\rho c)_f$ standing for the heat capacities of the solid and the interstitial fluid in the porous matrix. This criterion for the nonexistence of overstability is not only independent of the wave number of disturbances but also leads to the interesting conclusion that overstability may occur even if $\sigma > 1$ provided $S\varepsilon\sigma < 1$. This result is in marked contrast with that for a homogeneous fluid in the

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absence of a porous medium, where overstability can occur if $\sigma < 1$ Chandrasekhar [12].

The second part of the paper is an extension of the analysis in the first part to include finite amplitude disturbances for steady convection in the form of rolls. In order to explore the possibility of such convection at subcritical values of the Rayleigh number R we employ, following Veronis [13], a truncated representation of various physical variables (e.g., temperature, velocity, etc.) with amplitudes as functions of time. This representation takes into account the distortion of the temperature and velocity field due to non-linear interaction in a limited manner. The analysis reveals that steady convection at a subcritical values of R is possible for small values of the Taylor number Ta provided an inequality involving Ta , ε , σ , α and B holds, where α is the dimensionless wave number and $B = d^2/k$, k being the permeability of the layer of thickness d . However, for sufficiently large values of Ta , subcritical steady convection is not possible and overstable convection is found to be the preferred mode.

2. Mathematical Formulation

Consider a horizontal layer of a saturated porous medium of thickness d between the planes $z = 0$ and d , the z -axis being vertically upward. The layer rotates about the z -axis with uniform angular velocity Ω and is heated from below. The lower and upper boundaries are maintained at temperature T_0 and $T_0 - \Delta T$, respectively. The temperature T_{total} in the perturbed state is taken as

$$T_{\text{total}} = T_t = T_0 - \Delta T(z/d) + T(x, z, t), \quad (1)$$

where $T(x, z, t)$ is the deviation of the temperature from the linear profile, and we assume that $T(x, z, t)$ contains a non-zero horizontal mean.

The general form of the Darcy-Oberbeck-Boussinesq equation in a frame of reference rotating with angular velocity Ω is

$$\frac{1}{\varepsilon} \frac{D\mathbf{q}}{dt} = -\frac{1}{\varrho_0} \nabla p - \frac{\mu}{\varrho_0 k} \mathbf{q} + \frac{\mu}{\varrho_0} \nabla^2 \mathbf{q} - 2\Omega \times \mathbf{q} + g\beta T e_z, \quad (2)$$

where ϱ_0 , \mathbf{q} , p , μ , g and β denote fluid density, filtration velocity, pressure, dynamic viscosity coefficient,

acceleration due to gravity and thermal expansion coefficient, respectively. For very fluffy foam metal materials or fibrous materials, ε is very close to unity, and in beds of packed spheres, ε is in the range of 0.25–0.50 (Joseph [14]). The viscous term $\mu \nabla^2 \mathbf{q}$ is taken into account for the general flow when the fluid occupies the major part of the porous medium. The equation of continuity is

$$\nabla \cdot \mathbf{q} = 0, \quad (3)$$

and the energy equation due to Caltagirone [15] is

$$(\varrho c)^* \frac{\partial T_t}{\partial t} + (\varrho c)_f \mathbf{q} \cdot \nabla T_t = \lambda^* \nabla^2 T_t, \quad (4)$$

where the velocity \mathbf{q} has the components (u, v, w) .

The porous medium consisting of the porous matrix and the interstitial fluid (fluid in the pores) is regarded as a fictitious isotropic fluid with heat capacity $(\varrho c)^* = \varepsilon (\varrho c)_f + (1 - \varepsilon) (\varrho c)_s$. We assume that the Boussinesq approximation holds and the effective thermal conductivity λ^* (defined in the same manner as $(\varrho c)^*$ as given earlier) is constant. Using (1) in (4), the energy equation reduces to

$$(\varrho c)^* \frac{\partial T}{\partial t} - (\varrho c)_f (\Delta T/d) w + (\varrho c)_f \mathbf{q} \cdot \nabla T = \lambda^* \nabla^2 T. \quad (5)$$

We now assume that the convective motion occurs as a two-dimensional pattern of rolls. Such an assumption is justified due to the experimental observation by Koschmieder [16] that the preferred cellular pattern of convection generally consists of two-dimensional rolls whose orientation is determined by lateral boundaries.

If we assume that the y -axis is aligned along the rolls, then the physical variables are independent of y . The equation of continuity (3) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

which implies the existence of a stream function $\psi(x, z, t)$ such that $u = \partial \psi / \partial z$, $w = -\partial \psi / \partial x$.

Note that due to coriolis forces in the presence of rotation, a component of velocity $v(x, z, t)$ along the y -axis is induced.

We now introduce the dimensionless quantities

$$\mathbf{q}' = \mathbf{q} d / \lambda, \quad \tau = \lambda t / d^2, \quad (x', y', z') = (x, y, z) / d, \quad T' = T / \Delta T, \quad S = (\varrho c)^* / (\varrho c)_f, \quad \psi' = \psi / \lambda. \quad (6)$$

Using (6) with $u = \partial\psi/\partial z$, $w = -\partial\psi/\partial x$ in (2) and (5) and then dropping the primes, the dimensionless form of the momentum and energy equations becomes

$$\left(\frac{1}{\varepsilon} \frac{\partial}{\partial \tau} + \sigma B - \sigma \nabla^2\right) \nabla^2 \psi = \frac{1}{\varepsilon} J(\psi, \nabla^2 \psi) + \sigma (Ta)^{1/2} \partial v / \partial z - \sigma R \partial T / \partial x, \quad (7)$$

$$\left(\frac{1}{\varepsilon} \frac{\partial}{\partial \tau} + \sigma B - \sigma \nabla^2\right) v = \frac{1}{\varepsilon} J(\psi, v) - \sigma (Ta)^{1/2} \partial \psi / \partial z, \quad (8)$$

$$S(\partial T / \partial \tau) = J(\psi, T) - \partial \psi / \partial x + \nabla^2 T, \quad (9)$$

where J stands for the Jacobian and the dimensionless parameters Ta , R , σ , and B are given by

$$Ta = 4\Omega^2 d^4 / \nu^2, \quad \sigma = \nu / \lambda, \\ R = g \beta \Delta T d^3 / \lambda \nu, \quad B = d^2 / k, \quad (10)$$

where $\lambda = \lambda^* / (\rho c)_f$. Here Ta , σ , R , and B stand for the Taylor number, Prandtl number, Rayleigh number and the permeability parameter, respectively. For the sake of mathematical simplicity, we assume the boundaries $z = 0$ and 1 to be stress-free and perfect conductors of heat. So the boundary conditions are

$$T = \psi = \partial^2 \psi / \partial z^2 = \partial v / \partial z = 0 \text{ on } z = 0 \text{ and } 1. \quad (11)$$

3. Stability Analysis

We now consider the following three aspects of the stability problem for which the mathematical formulation is given in Sect. 2:

(i) Stationary convection, (ii) Time-dependent convection and (iii) Finite-amplitude convection. Our aim is to determine the critical Rayleigh number (corresponding to onset of convection) in these three cases. However, the cases (i) and (ii) are to be studied within the framework of linear stability theory.

(i) Linear Stability Analysis for Stationary Convection

For the investigation of linear stability, the non-linear terms contained in the Jacobians in (7)–(9)

are neglected. Further, when the onset of convection is stationary we may set $\partial/\partial\tau = 0$, so that (7)–(9) reduce to

$$\nabla^4 \psi + (Ta)^{1/2} \partial v / \partial z - B \nabla^2 \psi - R \partial T / \partial x = 0, \quad (12)$$

$$\nabla^2 v - (Ta)^{1/2} \partial \psi / \partial z - B v = 0, \quad (13)$$

$$\nabla^2 T - \partial \psi / \partial x = 0. \quad (14)$$

We now make a normal mode analysis of (12)–(14) (Chandrasekhar [12]) and assume the modal solutions consistent with the boundary conditions (11) as

$$\begin{aligned} \psi &= p \sin \pi \alpha x \sin n \pi z, \\ T &= [-p / (\pi^2 (\alpha^2 + n^2))] \cos \pi \alpha x \sin n \pi z, \\ v &= -[p (Ta)^{1/2} n \pi / (\pi^2 (\alpha^2 + n^2) + B)] \\ &\quad \cdot \sin \pi \alpha x \cos n \pi z, \end{aligned} \quad (15a, b, c)$$

where $p = \pi^2 (1 + \alpha^2)$.

Substitution of (15) in (12)–(14) leads to

$$R = \frac{\pi^2 (\alpha^2 + n^2) [(\alpha^2 + n^2) \{\pi^2 (\alpha^2 + n^2) + B\}^2 + Ta n^2]}{\alpha^2 [\pi^2 (\alpha^2 + n^2) + B]}. \quad (16)$$

Setting

$$\begin{aligned} \alpha^2 &= x, \quad R_s = R / \pi^4, \quad B_1 = B / \pi^2 \\ \text{and } T_1 &= Ta / \pi^4 \end{aligned} \quad (17)$$

in (16), the characteristic value R_s for the steady marginal state corresponding to the lowest mode $n = 1$ becomes

$$R_s = \frac{(1+x) [(1+x)(1+x+B_1)^2 + T_1]}{x(1+x+B_1)}. \quad (18)$$

This gives R_s as a function of the wave number x for given values of B_1 and T_1 . The minimum of R_s with respect to x gives the critical Rayleigh number R_s^c corresponding to steady convection.

(ii) Linear Stability for the Time-dependent Convection

We now explore the possibility of instability occurring in the form of an oscillation of increasing amplitude, i.e. as overstability. The system of equations in this case can be obtained from (7)–(9) by setting the non-linear parts, i.e. the terms containing the Jacobians, equal to zero; then they become

$$\begin{aligned} \left(\frac{1}{\varepsilon} \frac{\partial}{\partial \tau} + \sigma B - \sigma \nabla^2\right) \nabla^2 \psi \\ = \sigma (Ta)^{1/2} \partial v / \partial z - R \sigma \partial T / \partial x, \end{aligned} \quad (19)$$

$$\left(\frac{1}{\varepsilon} \frac{\partial}{\partial \tau} + \sigma B - \sigma \nabla^2\right) v = -\sigma (Ta)^{1/2} \partial \psi / \partial z, \quad (20)$$

and

$$S(\partial T / \partial \tau) = -\partial \psi / \partial x + \nabla^2 T. \quad (21)$$

After eliminating v and T from (19)–(21) the equation for ψ becomes

$$\begin{aligned} & \left(S \frac{\partial}{\partial \tau} - \nabla^2\right) \\ & \cdot \left\{ \left[\frac{1}{\varepsilon} \frac{\partial}{\partial \tau} - \sigma \nabla^2 + \sigma B \right]^2 \nabla^2 + \sigma^2 Ta D^2 \right\} \psi \\ & = \sigma R \left(\frac{1}{\varepsilon} \frac{\partial}{\partial \tau} - \sigma \nabla^2 + \sigma B \right) \frac{\partial^2 \psi}{\partial x^2}, \end{aligned} \quad (22)$$

where D stands for $\partial / \partial z$.

We assume that the solution for ψ is of the form

$$\psi = A e^{im\tau} \sin \pi \alpha x \sin n \pi z, \quad (23)$$

which satisfies the boundary conditions (11).

Using (23) in (22), we get

$$R = \frac{(ims + Y)}{\sigma \alpha^2} \left[\left(\frac{im}{\varepsilon} + X \right) (\alpha^2 + n^2) + \frac{\sigma^2 n^2 Ta}{\left(\frac{im}{\varepsilon} + X \right)} \right], \quad (24)$$

where

$$X = \sigma \pi^2 (\alpha^2 + n^2) + \sigma B \quad \text{and} \quad Y = \pi^2 (\alpha^2 + n^2). \quad (25)$$

Since we are interested in specifying the critical Rayleigh number for the onset of instability via a state of purely oscillatory motions, it will be sufficient for us to consider only those conditions which will give the solutions of (24) for which m will be real in the marginal state. Following [12], we equate the real and imaginary parts of (24); for the lowest mode ($n = 1$) these reduce to

$$R = \frac{(1 + \alpha^2)}{\sigma \alpha^2} \cdot \left[\left(XY - \frac{m^2 s}{\varepsilon} \right) + \frac{Ta \sigma^2}{(1 + \alpha^2)} \left(\frac{XY + \frac{m^2 s}{\varepsilon}}{X^2 + \frac{m^2}{\varepsilon^2}} \right) \right] \quad (26)$$

and

$$(1 + \alpha^2) \cdot \left[\left(SX + \frac{Y}{\varepsilon} \right) + \frac{Ta \sigma^2}{(1 + \alpha^2)} \left(\frac{SX - \frac{Y}{\varepsilon}}{X^2 + (m^2/\varepsilon^2)} \right) \right] = 0, \quad (27)$$

where X and Y are given by (25) with $n = 1$.

One can write

$$\frac{\sigma^2 Ta}{(1 + \alpha^2)} \left(\frac{XY + \frac{m^2 s}{\varepsilon}}{X^2 + (m^2/\varepsilon^2)} \right) \quad (28)$$

$$= \frac{\sigma^2 Ta S \varepsilon}{(1 + \alpha^2)} - \frac{\varepsilon X \sigma^2 Ta \left(SX - \frac{Y}{\varepsilon} \right)}{(1 + \alpha^2) (X^2 + (m^2/\varepsilon^2))}.$$

From (27), we have

$$- \frac{\sigma^2 Ta}{(1 + \alpha^2)} \left(\frac{SX - \frac{Y}{\varepsilon}}{X^2 + \frac{m^2}{\varepsilon^2}} \right) \varepsilon X = \varepsilon X \left(SX + \frac{Y}{\varepsilon} \right). \quad (29)$$

Using (29) in (28) we have

$$\begin{aligned} & \frac{\sigma^2 Ta}{(1 + \alpha^2)} \left(\frac{XY + \frac{m^2 s}{\varepsilon}}{X^2 + \frac{m^2}{\varepsilon^2}} \right) \\ & = \frac{S \varepsilon \sigma^2 Ta}{(1 + \alpha^2)} + \varepsilon X \left(SX + \frac{Y}{\varepsilon} \right). \end{aligned} \quad (30)$$

Solving (27) for m^2/ε , we find

$$\frac{m^2}{\varepsilon} = \frac{\sigma^2 Ta \left(\frac{Y}{\varepsilon} - SX \right) \varepsilon}{(1 + \alpha^2) \left(SX + \frac{Y}{\varepsilon} \right)} - \varepsilon X^2. \quad (31)$$

Using (25) (with $n = 1$), (30), and (31) in (26), we get

$$R_0 = \frac{2(1 + x + B_1)}{x} \cdot \left[\frac{(1 + x) \{ (1 + x) (1 + S \varepsilon \sigma) + S \varepsilon \sigma B_1 \}^2 + T_1 \sigma^2 S^2 \varepsilon^2}{(1 + x) (1 + S \varepsilon \sigma) + S \varepsilon \sigma B_1} \right], \quad (32)$$

where $R_0 = R/\pi^4$ and B_1 , T_1 and x are defined in (17).

When $S = \varepsilon = 1$ and $B_1 = 0$, the expression for R_0 in (32) reduces to the corresponding value of R_0 for thermal instability in a non-porous medium with rotation (Chandrasekhar [12]). From (32), the minimum of R_0 with respect to x can be found for given values of S , ε , σ , B_1 , and T_1 . This minimum, denoted by R_0^c , gives the critical Rayleigh number for overstable oscillations.

From (31) one can conclude that overstability can occur, i.e. $m^2 > 0$, if

$$\frac{\sigma^2 \text{Ta} \left(\frac{Y}{\varepsilon} - SX \right)}{(1 + \alpha^2) \left(SX + \frac{Y}{\varepsilon} \right)} > X^2. \quad (33)$$

After substituting from (25) with $n = 1$ in (33), one finds that overstability occurs if

$$\frac{T_1[(1+x)(1 - S\varepsilon\sigma) - S\varepsilon\sigma B_1]}{(1+x)[(1+x)(1 + S\varepsilon\sigma) + S\varepsilon\sigma B_1]} > (1+x+B_1)^2. \quad (34)$$

Clearly the above inequality can not hold if $S\varepsilon\sigma > 1$ since S , ε , σ , and B_1 are all positive. Hence overstability cannot occur if $S\varepsilon\sigma > 1$, and hence in this case the principle of exchange of stabilities holds. It can be readily seen from (34) that for overstability to be at all possible, we must have $S\varepsilon\sigma < 1$. Even in this case, real values of frequency will exist if

$$T_1 > (1+x)(1+x+B_1)^2 \cdot \left[\frac{(1+x)(1 + S\varepsilon\sigma) + S\varepsilon\sigma B_1}{(1+x)(1 - S\varepsilon\sigma) - S\varepsilon\sigma B_1} \right]. \quad (35)$$

It is interesting to note that the necessary condition $S\varepsilon\sigma < 1$ for the existence of overstability can be fulfilled even for a fluid with $\sigma > 1$. This is in marked contrast with the case of a homogeneous fluid in the absence of a porous medium (Chandrasekhar [12]), where overstability can occur only if $\sigma < 1$.

(iii) Finite Amplitude Convection

We now consider the possibility of onset of instability as convection with finite amplitude at subcritical values of R , i.e. for $R < R_c$, where R_c is the critical Rayleigh number according to linear theory. As already mentioned in Sect. 2, we shall assume that convection sets in in the form of two-dimensional rolls with their axis parallel to the y -direction.

Our analysis is now based, following Veronis [13], on a truncated representation of the finite amplitude convection with a view to obtaining certain physical results of significance.

It can be seen from (8) that the first effect of non-linearity is to distort the zonal velocity v through the nonlinear interaction of ψ and v given by the Jacobian term. Similarly, from (9) it can be seen

that there is a distortion of the temperature field through the interaction between ψ and T reflected in the Jacobian $J(\psi, T)$. Thus it can be readily shown that, due to the above distortion of the temperature field, a component of the form $\sin 2\pi z$ is generated. Similarly, for the zonal velocity v , the distortion leads to a component of the form $\sin 2\pi\alpha x$.

Hence we take a limited representation of the finite amplitude convection in the form

$$\psi = 2 \left(\frac{p}{2} \right)^{1/2} a(\tau) \sin \pi \alpha x \sin \pi z, \quad (36)$$

$$T = -2 \left(\frac{1}{2p} \right)^{1/2} \pi \alpha b(\tau) \cos \pi \alpha x \sin \pi z - \pi^3 \alpha^2 c(\tau) \sin 2\pi z, \quad (37)$$

$$v = - \frac{2\pi \text{Ta} \left(\frac{p}{2} \right)^{1/2}}{p+B} d(\tau) \sin \pi \alpha x \cos \pi z + \frac{p \pi^3 \text{Ta}}{(p+B)} e(\tau) \sin 2\pi \alpha x, \quad (38)$$

where $p = \pi^2(1 + \alpha^2)$, as defined before, and the modal amplitudes a, b, c, d , and e are generally functions of time. Substituting (36)–(38) in (7)–(9) and equating the coefficients of $\sin \pi \alpha x \sin \pi z$, $\sin \pi \alpha x \cos \pi z$, etc., we obtain the following set of equations for the amplitudes:

$$\dot{a} = \frac{\sigma R \varepsilon (\pi \alpha)^2}{p^2} b - \frac{\sigma \varepsilon \pi^2 \text{Ta}}{p(p+B)} d - \sigma \varepsilon (p+B) a, \quad (39)$$

$$\dot{b} = \frac{2p \pi^4 \alpha^2}{S} a c + \frac{p}{S} a - \frac{p}{S} b, \quad (40)$$

$$\dot{c} = \frac{a b}{S} - \frac{4\pi^2}{S} c, \quad (41)$$

$$\dot{d} = 2p \pi^4 \alpha^2 a e + \sigma \varepsilon (p+B) a - \sigma \varepsilon (p+B) d, \quad (42)$$

$$\dot{e} = a d - \sigma \varepsilon (B + 4\pi^2 \alpha^2) e, \quad (43)$$

where an overdot denotes derivative with respect to time.

The Eqs. (39)–(43) are a set of ordinary nonlinear differential equations and are too complicated to be amenable to analytical solution. To draw some useful informations, we assume the steady state solutions of the above system of equations. Equa-

tions (40)–(43) with $\dot{a} = \dot{b} = \dots = 0$, lead to the relations

$$c = \frac{ab}{4\pi^2}, \quad (44)$$

$$e = a d / [\sigma \varepsilon (B + 4\pi^2 \alpha^2)], \quad (45)$$

$$d = \frac{(\sigma \varepsilon)^2 (p + B) (B + 4\pi^2 \alpha^2) a}{(\sigma \varepsilon)^2 (p + B) (B + 4\pi^2 \alpha^2) - 2p \pi^4 \alpha^2 a^2}, \quad (46)$$

$$b = a \left/ \left(1 - \frac{\pi^2 \alpha^2}{2} a^2 \right) \right. \quad (47)$$

Using (46) and (47) in (39) with $\dot{a} = 0$, the equation for the amplitude a in the steady state is given by

$$\left[p^3 \pi^4 \alpha^2 a^4 + \left(\frac{2\pi^4 p^2 \text{Ta}}{(p + B)^2} - \frac{Z \pi^2 \alpha^2}{2} R_c \right) a^2 + Z(R_c - R) \right] a = 0, \quad (48)$$

where $Z = (\varepsilon \sigma)^2 (B + 4\pi^2 \alpha^2)$. Now (48) gives $a = 0$, which corresponds to pure conduction. The remaining solutions of (48) with $a \neq 0$ corresponds to steady convective motion and are given by

$$a^2 = \left\{ \left[\frac{\alpha^2 \pi^2 (p + B)}{p} (R - R_c) - \frac{Z \pi^2 \alpha^2 (p + B)^2}{4p^2} R_c + \pi^4 \text{Ta} \right] \pm \left[\left[\frac{\alpha^2 \pi^2 (p + B)}{p} (R - R_c) - \frac{Z \pi^2 \alpha^2 (p + B)^2}{4p^2} R_c + \pi^4 \text{Ta} \right]^2 + \frac{Z \pi^4 \alpha^2 (p + B)^4}{p} (R - R_c) \right]^{1/2} \right\} / p \pi^4 \alpha^2 (p + B)^2. \quad (49)$$

However, the positive sign in front of the square root in (49) is admissible since otherwise a^2 becomes negative, which gives an imaginary amplitude. We next consider the possibility of the existence of finite amplitude solutions with $R < R_c$. The minimum value of R for which the subcritical solutions will exist may be obtained for those values of R which will make the discriminant in (49) vanish provided the first term on the right hand side of

(49) is non-negative. It can be readily shown that the discriminant in (49) vanishes for two values of R of which one value leads to an imaginary value of a (and hence inadmissible) while the other value is R_f where

$$R_f = \left[\{(1 + x)(1 + x + B_1) \cdot [4(1 + x) - (4x^2 + 4x + 5xB_1 + B_1 + B_1^2) \varepsilon^2 \sigma^2]\}^{1/2} + \sigma \varepsilon \{T_1(B_1 + 4x)\}^{1/2} \right] / 4x. \quad (50)$$

The minimum value of R_f with respect to x gives the critical Rayleigh number R_f^c for steady finite amplitude convection for given values of ε , σ , B_1 and T_1 .

Now (50) holds only if the first term on the right hand side of (49) is non-negative, which gives

$$T_1 [(1 + x) \{4(1 - x \varepsilon^2 \sigma^2) - \varepsilon^2 \sigma^2 B_1\} - \varepsilon^2 \sigma^2 B_1 (B_1 + 4x)] > \varepsilon^2 \sigma^2 (1 + x) (4x + B_1) (1 + x + B_1)^3. \quad (51)$$

It can be clearly seen from (51) that this inequality is meaningful if $x \varepsilon^2 \sigma^2 < 1$.

4. Discussion

Figure 1 shows the variation of the critical Rayleigh number R_s^c determined from (18) with T_1 for different values of B_1 . It is seen that for small values of T_1 , R_s^c increases with B_1 . Since $B_1 = d^2/k$, we conclude that for small T_1 , permeability is destabilizing. But for large T_1 the effect of permeability is stabilizing. Figure 2 shows the variation of R_s^c with T_1 for different values of ε and B_1 , S and σ being held fixed. It can be seen that R_s^c increases with either ε or B_1 , which means that while porosity is stabilizing, the permeability exerts a destabilizing influence on the flow. Further it can be seen that the onset of overstable convection is delayed with increase in B_1 , i.e. with decrease in permeability. Figure 3 shows for small T_1 , ε has no effect on R_f^c whereas for large T_1 , the effect of B_1 on R_f^c is negligible. Figure 4 shows that, as long as T_1 is not very large, finite amplitude steady motion can exist for values of R_f^c smaller than R_s^c . But for large values of T_1 , $R_s^c < R_f^c$ and hence the overstable mode is preferred in this case. The physical reason for this is that finite amplitude motions can occur for subcritical values of R as long as these motions can

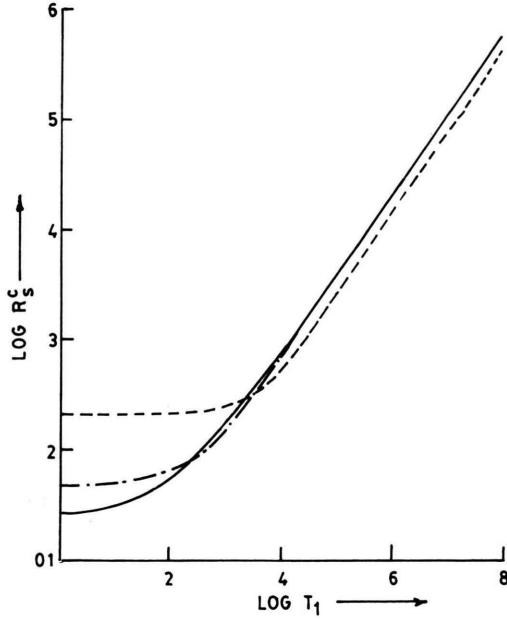


Fig. 1. Variation of the critical Rayleigh number for stationary convection R_s^c with Taylor number T_1 for different values of the permeability parameter B_1 . Solid line, for $B_1 = 5$, dashed line for $B_1 = 10$ and dash-dotted for $B_1 = 50$, respectively.

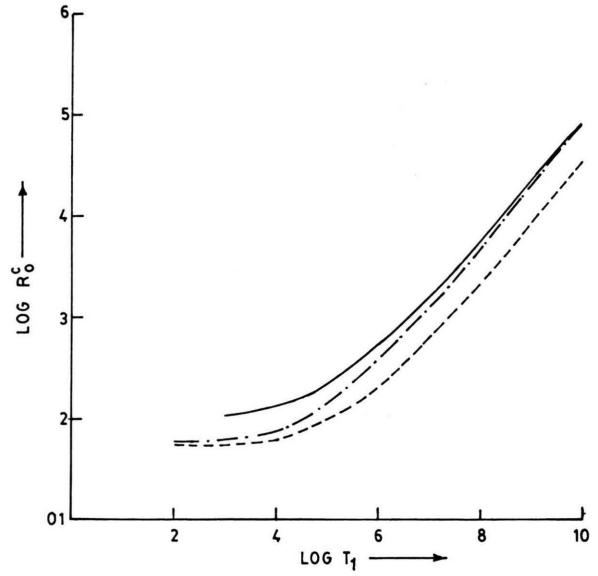


Fig. 2. Variation of the critical Rayleigh number for time-dependent convection R_0^c with Taylor number T_1 for different values of porosity ϵ and B_1 with $\sigma = 0.025$ and $S = 0.9$. $\epsilon = 0.8$, $B_1 = 10$, solid line, $\epsilon = 0.8$, $B_1 = 5$, dash-dotted and $\epsilon = 0.4$, $B_1 = 5$, dashed line.

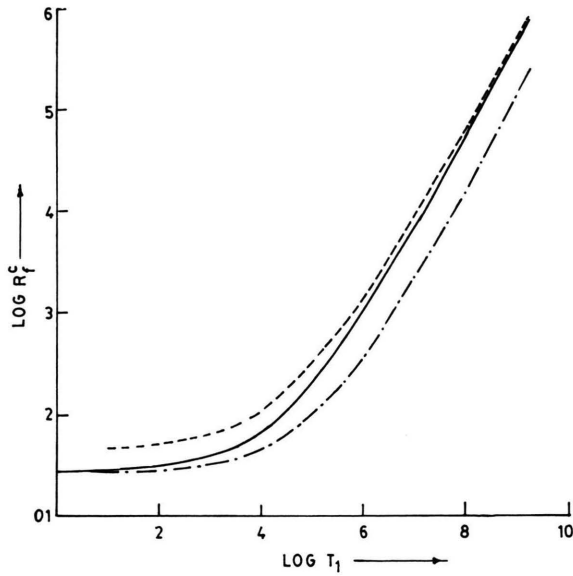


Fig. 3. Variation of the critical Rayleigh number for finite amplitude convection R_f^c with Taylor number T_1 for different values of ϵ and B_1 and $\sigma = 0.025$ kept fixed. $\epsilon = 0.8$, $B_1 = 5$, solid line, $\epsilon = 0.8$, $B_1 = 10$, dashed line, $\epsilon = 0.4$, $B_1 = 5$, dash-dotted.

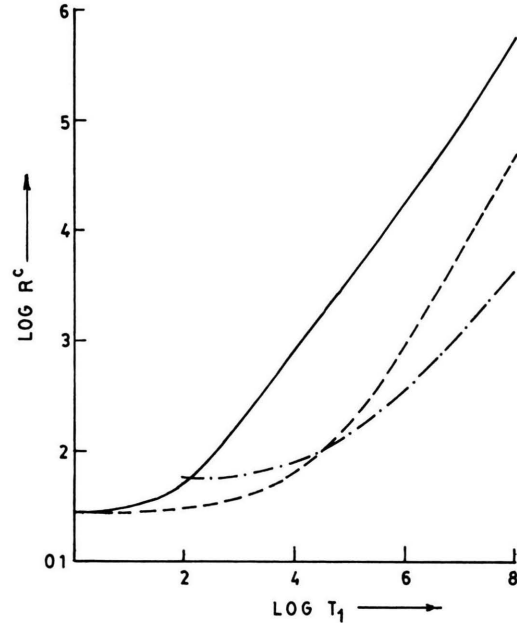


Fig. 4. Variation of R_s^c , R_0^c and R_f^c with T_1 for $\sigma = 0.025$, $\epsilon = 0.8$, $B_1 = 5$ and $S = 0.9$. R_s^c , R_0^c and R_f^c curves represent solid, dash-dotted and dashed line, respectively.

offset the constraint of rotation. Hence for large values of T_1 the motion must have a large amplitude to offset the constraint of rotation. But greater amplitudes require larger release of potential energy which in turn implies a larger value of R . Thus it is expected that finite amplitude motion is unlikely to occur for large T_1 , and overstability becomes the preferred mode. However, it should be pointed out that the existence of oscillatory subcritical instabil-

ity can not be ruled out, and this aspect will form the subject matter of a future study.

Acknowledgement

The author expresses his sincere thanks to Professor A. S. Gupta for his valuable comments on this paper.

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